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TOWARDS A MEANINGFUL INTRODUCTION TO THE SCHRÖDINGER EQUATION THROUGH HISTORICAL AND HEURISTIC APPROACHES

ABSTRACT: The Schrödinger equation is the foundation of quantum mechanics and the starting point for any improvement to the description of submicroscopic physical and chemical systems. Although it cannot be proved or derived strictly, it has associated with it various formulations and 'derivations'. In this work, we study the physical and mathematical evidence that make the Schrödinger equation plausible. First, we make an approach from the historical perspective, and study the methods of the pioneers Schrödinger, Heisenberg, and Dirac. Turning to modern treatments, we review numerous heuristic introductions to quantum mechanics and make a synthesis of the various methods into a coherent and meaningful whole. [*Chem. Educ. Res. Pract. Eur.*: 2001, *2*, 203-213]

INTRODUCTION

Quantum mechanics was introduced first by Heisenberg in 1925 and in the same year given the matrix-algebra formulation by Born and Jordan. One year later (1926), Schrödinger independently introduced his wave mechanics and proved the equivalence of the two methods. In classical mechanics, there are various formulations of the fundamental equations, such as Newton's laws of motion, Lagrange's equations, and Hamilton's equations. The same is true in quantum mechanics: Schrödinger's equation of motion led to wave mechanics, while Heisenberg's equation of motion led to matrix mechanics.

Dirac gave a general formulation of quantum mechanics, in which Schrödinger's and Heisenberg's pictures appear as special cases. In addition, Dirac examined the conditions under which the two methods are equivalent. Finally, the connection between matrix mechanics and wave mechanics was made clear by the mathematician von Neumann in the early 1930s. Von Neumann promoted the wavefunctions and their properties to postulates. In this approach, the Schrödinger equation, together with the properties of its solutions, is a fundamental law of nature, whose validity is confirmed only by experiment. Baggot (1992) provides a historical account of the old quantum theory and of quantum mechanics, and discusses the conceptual meaning of quantum theory.

The Schrödinger equation cannot be proved or derived strictly. Schrödinger was led to it from the analogies between geometrical optics and classical mechanics, and between

undulatory optics and wave mechanics. The non-rigorous derivations of the Schrödinger equation, together with the mathematical and conceptual complexities, may be one reason why the postulative approach is very popular, especially with chemists, who are interested in the applications of quantum mechanics to chemistry. However, there are many ways to introduce the equation, both from the historical perspective and by heuristic treatments, that provide insights into the meaning of quantum mechanics.

According to Lakatos (1970), science progresses through competing theories (competing research programs). This methodology has further been applied to the reconstruction of students' and teachers' understanding of science content (Blanco & Niaz, 1998), since it is considered that a parallel exists between the historical development of a theory and an individual's acquisition of knowledge (Blanco & Niaz, 1998; Duschl & Gitomer, 1991; Kitchener, 1987; Piaget & Garcia, 1989, von Glasserfeld, 1989). On the contrary, the presentation of science as a corpus of perfect, final products and postulates is an obstacle to conceptual understanding. Telling the 'facts' - the postulates - is not likely to lead to a genuine understanding of quantum mechanics, just as telling that matter is composed of particles separated by empty space does not lead to an understanding of vacuum and the corpuscular nature of matter (Nussbaum, 1997). It is imperative then that we delve into the varieties of introducing quantum mechanics.

In this work we refer to the physical and mathematical evidence that make the Schrödinger equation plausible. First, we make a diversion to the history: the methods of the pioneers Schrödinger, Heisenberg, and Dirac. Then, we review and synthesize various heuristic introductions to quantum mechanics. At the outset, it must be stressed that the following account is directed mainly at university teachers who are engaged in teaching courses on quantum mechanics or quantum chemistry. It is not meant to be an outline of how such a course should be taught.

1. THE METHODS OF THE PIONEERS SCHRÖDINGER, HEISENBERG AND DIRAC

1.1. Schrödinger's equation of motion: Wave mechanics

Though it occurred after Heisenberg's, we begin with Schrödinger's method because it is conceptually and mathematically simpler. Schrödinger (1926a, 1926b) started with the research of de Broglie (1924, 1925) on matter waves, and concluded that the non-applicability of classical mechanics to micro-mechanical problems is perhaps of exactly the same kind as the non-applicability of geometrical optics to the phenomena of diffraction or interference and may, perhaps, be overcome in a similar way. He started from the classical wave equation,

$$\nabla^2 \Psi - \frac{1}{v_\varphi^2}\frac{\partial^2 \Psi}{\partial t^2} = 0 \,, \tag{1}$$

where v_φ is the phase velocity of the wave that describes the motion of the particle, and found that the functions Ψ can depend on time only through the factor $\exp(\pm iEt/\hbar)$, thus writing

$$\Psi(\vec{r},t) = \psi(\vec{r})e^{\pm iEt/\hbar} \,. \tag{2}$$

In this way, the time-independent Schrödinger equation, $\hat{H}\psi = E\psi$ resulted. Further, taking the time derivative of equation (2), and eliminating E, he arrived at the required time-dependent Schrödinger equation:

$$\hat{H}\psi = \pm\frac{\hbar}{i}\frac{\partial\Psi}{\partial t}.$$

(E had to be eliminated because the final equation must hold in any case, whatever the dependence of Ψ on time.)

For the derivation of the time-independent quantum wave equation, Schrödinger used also an integral-variation principle, by applying the calculus of variations (Margenau & Murphy, 1956, pp. 213-215), with E being the Lagrange multiplier, while the extremum condition is the well-known variation theorem. This derivation led Schrödinger to extend the variation problem, and discover the time-independent and time-dependent wave equations for a wholly arbitrary conservative system.

1.2. Heisenberg's equation of motion: Matrix mechanics

In his attempt to formulate quantum mechanics, Heisenberg was guided by the fundamental principle that only observable quantities should enter the theory. Quantities like the position and the velocity of the electron are not observable, and hence were banished from the theory [the observability principle (Heisenberg, 1925; Mehra & Rechenberg, 1982)].

Heisenberg was convinced that there should be quantum mechanical analogues of the Fourier series. A central question for him was: "Can one multiply two quantum amplitudes X and Y, which in classical physics correspond to Fourier series?" He provided a rule for this multiplication, which was different for the multiplication of two Fourier series describing the variables of classical periodic systems (Pipes, 1958). Born and Jordan (1925) connected Heisenberg's multiplication rule with matrix algebra and discovered the occurrence of two non-commuting matrices in Heisenberg's quantum condition:

$$\mathbf{pq} - \mathbf{qp} = \frac{\hbar}{i}\mathbf{1}, \tag{3}$$

where $\mathbf{1}$ stands for the infinite unit matrix. Thus, the classical co-ordinates q_i and momenta p_i are to be replaced by Hermitian matrices (in general infinite) such that equation (3) is satisfied. From these matrices one then constructs the Hamiltonian matrix, $\mathbf{H(q_i, p_i)}$, which satisfies the classical Hamilton equations of motion. These equations led Born and Jordan to conclude that for any analytic function $\mathbf{g(p, q)}$, the time derivative is given by the equation

$$d\mathbf{g}/dt = \frac{i}{\hbar}(\mathbf{Hg} - \mathbf{gH}).$$

Taking $\mathbf{g} = \mathbf{H}$, we obtain $d\mathbf{H}/dt = 0$, that is, the energy is conserved for all quantum systems with separable Hamiltonian. Finally, when the Hamiltonian matrix is put into diagonal form by a unitary transformation, the diagonal elements are the possible energies of the system.

1.3. Dirac's general formulation of quantum mechanics

Dirac in his classic book (Dirac, 1958, pp. 1-135) presents quantum mechanics through transformation theory. This symbolic method, as opposed to the method of coordinates or representations, deals in an abstract way with the quantities of interest. Schrödinger's wave mechanics and Heisenberg's matrix mechanics appear in this theory as special cases. Dirac started with the principle of superposition, and distinguished between *large,* which is not affected by observation, and *small*, which is perturbed by observation; the large is treated by classical mechanics, while the small needs the new quantum mechanics. When one makes an observation on a dynamical system, the state of the system changes in an unpredictable way, but in between observations causality applies to quantum mechanics as in classical mechanics; the system then is governed by equations of motion that make the state at one time determine the state at a later time.

Dirac arrived at the time-dependent Schrödinger equation

$$\hat{H}(t)\,\big|\Psi\!\left(\xi t\right)\big\rangle = i\hbar\,\frac{d\big|\Psi\!\left(\xi t\right)\big\rangle}{dt},$$

with $\big|\Psi\!\left(\xi t\right)\big\rangle$ being the time dependent wave function (with ξ standing for the coordinates). This is the Schrödinger form of the equation of motion (the Schrödinger picture); according to this, the physical condition of a dynamical system at any time involves the relation of the dynamical variables to the state, and the change in the physical condition with time is ascribed to a change in a state, leaving the dynamical variables unchanged (fixed).

In the Heisenberg picture, the change of the physical condition with time is ascribed to a change in the dynamical variables, keeping the state fixed. Thus, a dynamical variable corresponding to the fixed linear operator $\hat{U}$ in the Schrödinger picture, in the Heisenberg picture corresponds to a moving linear operator $\hat{U}_t$, given by $\hat{U}_t = \hat{T}^{-1}\hat{U}\hat{T}$ or $\hat{T}\ \hat{U}_t = \hat{U}\hat{T}$. Differentiating the latter equation with respect to time, one can arrive at the following equation[1]

$$i\hbar\frac{d\hat{U}_t}{dt} = \hat{U}_t\hat{H}_t - \hat{H}_t\hat{U}_t, \qquad\qquad (4)$$

where $\hat{H}_t = \hat{T}^{-1}\hat{H}\,\hat{T}$. Equation (4) may be put in 'quantum bracket' notation (Landé, 1951, pp 107-150):

$$i\hbar\frac{d\hat{U}_t}{dt} = \left[\hat{U}_t - \hat{H}_t\right].$$

In this way, an analogy is established with the equations of motion of classical mechanics when the latter are written in the Poisson bracket notation. This analogy justifies the assumption that the linear operator $\hat{H}$ is the energy of the system in quantum mechanics.

Equation (4) shows how a dynamical variable changes with time in the Heisenberg picture and gives Heisenberg's form for the equation of motion.[2]

1.4. The equivalence of the Schrödinger and the Heisenberg method

Schrödinger's wave mechanics involves the solution of partial differential equations, while Heisenberg's matrix mechanics involves algebraic manipulations. Shortly after the publication of both theories, Schrödinger (1926c) showed that the two theories were identical. Let us summarize how Dirac treated this equivalence and the conditions under which such an equivalence does exist (Dirac, 1964).

Heisenberg introduced dynamical variables u which satisfy the Heisenberg equation of motion. The dynamical variables were first represented by matrices, but later they were considered as operators operating on to vectors of a Hilbert space. Denoting the Hilbert vector by $|A>$, the action of the dynamical variable u on $|A>$ is denoted by $u|A>$. Schrödinger gave emphasis to physical states represented by wave functions, which are also vectors $|A>$ in a Hilbert space. They differ from Heisenberg's Hilbert vectors in that they vary with time according to the Schrödinger wave equation. The Hamiltonians are the same operator in the two theories. In addition, the dynamical variables and the Hilbert vectors of the two theories are connected by a unitary transformation.

Dirac (1964) has shown however that the equivalence of the two theories is not complete. For instance, when we consider electron and positron interactions with the electromagnetic field, the Heisenberg equation of motion gives sensible field equations in agreement with classical electrodynamics; but the Schrödinger wave equation does not give any solution, not even approximate ones. From this it follows that the equivalence between the two theories may not be complete.

The mentioned unitary transformation does not exist for problems with infinite degrees of freedom. In this case it is the Heisenberg picture which is valid. However, in those problems in which only a finite number of degrees of freedom is effective, such as problems in which the electromagnetic interaction can be represented by Coulomb forces, the Schrödinger picture is equally valid and is used extensively, since it is simpler and more convenient. Dirac proposed further that the Heisenberg dynamical variables can no longer be taken to be operators in Hilbert space, but something more general. He called them q-numbers, which are non-commuting quantities, satisfying definite commutation relations.

Finally, the connection between matrix mechanics and wave mechanics was made clear by the mathematician von Neumann in the early 1930s (von Neumann, 1955), who expressed Schrödinger's wave mechanics in matrix form. Note that, while matrix mechanics depends on the properties of non-commuting matrices, wave mechanics can be derived from the properties of non-commuting operators (Baggot, 1992).

2. REVIEW AND SYNTHESIS OF HEURISTIC INTRODUCTIONS

As mentioned, Schrödinger's method is conceptually and mathematically simpler, so it gained general acceptance and application in most problems. It is adopted in almost all modern courses and textbooks on quantum physics and quantum chemistry.[3] Because the Schrödinger equation cannot be proved or derived strictly, some books adopt the postulative approach (e.g. Margenau & Murphy, 1956, pp. 337-344; Kauzmann, 1957; Sherwin, 1959; Matthews, 1968). On the other hand, many authors, especially in the literature of quantum chemistry, are mainly interested in the applications of quantum mechanics to the systems of their interest, for instance atoms and molecules.

According to the postulative approach, the Schrödinger equation together with the properties of its solutions are introduced through a number of postulates (the postulates of quantum mechanics), while no attempt is made to justify them physically, or at least to make them plausible. Instead the Schrödinger equation and the properties of its solutions are taken

as a fundamental law of nature, of which the validity is confirmed only by experiment. We have argued, however, at the introduction of this paper, that only an extended exposure to the variety of ways of introducing the Schrödinger equation can lead to a genuine understanding of the nature of quantum mechanics.

In the literature there are various methods of introduction to quantum mechanics, in accordance with the Schrödinger picture. Here we review the most common of them, and at the same time we attempt a synthesis into a coherent and meaningful whole. The number of books on quantum mechanics is vast. Of necessity, we will restrict ourselves only to some of them.

2.1. Wave packets and the equation of classical wave motion - The need for a quantum wave equation

Central to most modern introductions to quantum mechanics is the concept of a wave packet. This follows from de Broglie's hypothesis that a particle with momentum p could be replaced by a wave with wavelength given by the relation $\lambda = h/p$, or equivalently with propagation number $k = 2\pi/\lambda = p/\hbar$. The question then arises what kind of wave is to be considered as corresponding to the motion of a particle. The answer is the one-dimensional (plane) wave packet, that is a localized wave with zero amplitude everywhere except in a small region of space. This construct is a result of Fourier analysis (Bohm, 1951; Kemble, 1975, pp. 399-407).

The classical equation of wave motion [equation (1)] (Heitler, 1956; Pilar, 1968, pp. 35-44) is often taken as the starting point for deriving the Schrödinger equation. To account for quantum effects, a wave equation must fulfill a number of requirements (Schiff, 1968). Such an equation cannot strictly be derived. Its 'derivation' is based on some mathematical manipulations that are consistent with physical principles and the above requirements.

2.2. The time-dependent and the time-independent Schrödinger equations

The above discussion directs our attention to a derivation which is found, for instance, in Heitler (1956) and in Schiff (1968). The derivation starts with the classical wave equation, and arrives at the time-dependent Schrödinger equation. Note that the requirement that this equation is of the first order with respect to time is compatible with Heisenberg's uncertainty principle (Pilar, 1968, p. 53).

The equation $E = \hbar\omega = p^2/2m = \hbar^2 k^2/2m$, or equivalently the relation $\omega = \hbar k^2/2m$ which is used in the above derivation, is also the starting point of two other derivations (see e.g. Flurry, 1983; Bohm, 1951; Messiah, 1961, pp. 63-64).

The time-independent Schrödinger equation follows from the time-dependent Schrödinger equation by making the substitution $\Psi(x, t) = \psi(x)\varphi(t)$ in the latter and then separating the variables. A separation constant is introduced, and two equations result. The separation constant turns out to be equal to the energy of the system (Pilar, 1968, p. 65). In this way, the first equation is the time-independent Schrödinger equation. Finally, a derivation similar to the original derivation of Schrödinger, arriving first at the time-independent equation, can be found, for instance, in Fong (1962, pp. 42-47) and in Kemble (1975, pp. 14-19).

The extension to three dimensions is straightforward: instead of the second derivative with respect to x, we have the laplacian operator ∇^2. In addition, if the particle moves in a potential $V(r)$ which does not depend on time explicitly (conservative system), $V(r)$ has to be included in the hamiltonian operator $\hat{H} = (-\hbar^2/2m)\nabla^2 + V(r)$. The solutions of the one-

dimensional Schrödinger equation for the case in which *V* depends on position are not exactly of the plane wave form. This causes a conceptual difficulty. For a rigorous discussion, see Messiah (1961, pp. 65-67). Finally, for the quantum wave equation of a system of charged particles in a classical electromagnetic field, see Messiah (1961, pp. 67-68) or Kemble (1975, pp. 26-29).

2.3. The quantum mechanical momentum operator

The time-independent Schrödinger equation leads to the need for the replacement of classical momentum *p* by the quantum mechanical operator $-ih\nabla$. An entirely heuristic method of introducing this operator has been suggested by Chow (1992), and it may be an alternative method of introducing quantum mechanics at an elementary level, since it requires a minimum of mathematics. At the same time, the method offers an insightful conceptual approach to the fundamental difference between classical and quantum physics: while in classical physics any disturbance caused by the action of observation or measurement can, in principle, be calculated, in the submicroscopic world, the very action of observation or measurement of a dynamic variable disturbs the system (the uncertainty principle).

This fundamental difference leads to the *non-commutativity* of measurements on momentum and energy of quantum systems. As a consequence, while in classical physics the dynamical variables can be represented by ordinary algebraic variables, in quantum physics they are represented by matrices. The distinction between quantum (small) and classical systems is generally made in units of *h* or $\hbar$. The mutual disturbance of position and momentum measurements is thus expected to be related to *h*, so we write

$$[x, p] = xp - px = \alpha h,$$

where α is number to be determined. Using De Bloglie's proposition that a material particle, at a specific instance in time, could be represented with a spatial wave, leads to the identification that $\alpha = i$. In this way, the Schrödinger representations of the momentum and energy operators are obtained from the non-commutativity property of operators.

2.4. The Klein-Gordon and the Dirac relativistic equations

To take relativity into account, one has to use the equation $E^2 = p^2 c^2 + m_0^2 c^2$, which is consistent with the theory of relativity. From this, the Klein-Gordon relativistic wave equation is deduced (Messiah, 1961, p. 65; Mandl, 1957). This equation contains a second derivative with respect to time. In addition, in the non-relativistic limit of small kinetic energies, it reduces to the Schrödinger equation without spin. For this reason, the Klein-Gordon equation cannot be the correct relativistic equation for electrons. Dirac (1958, pp. 253-263) took the square-root of the expression for *E*, replaced in it *p* and *E* with the corresponding quantum-mechanical operators, and arrived at an equation that contains a first time derivative. Because the square-root operator that appears on the left-hand side of this equation has no meaning, he assumed the relationship

$$p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2 = \left(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m_0 c\right)^2,$$

and arrived at the following relativistic wave equation, the Dirac equation:

$$ic\hbar\left(\alpha_x\frac{\partial}{\partial x}+\alpha_y\frac{\partial}{\partial y}+\alpha_z\frac{\partial}{\partial z}\right)\Psi-\beta m_0 c^2\Psi=i\hbar\frac{\partial\Psi}{\partial t}\quad.$$

The α 's and β 's satisfy the following anticommutation relations: $\alpha_i\alpha_j=-\alpha_j\alpha_i$ (i ≠j); $\alpha_i\beta=-\beta\alpha_i$; $\alpha_i{}^2=\beta^2=1$. Dirac showed that the α 's and the β 's have to be 4×4 matrices and Ψ a four-component vector. Thus, the wave equation has several components and these are connected with spin (Dirac, 1958, pp. 253-263; Merchbacher, 1970; Landé, 1951, pp. 240-63; McKelvey, 1983). (In the last reference, there is also an interesting discussion of relativistic effects on chemical properties.)

2.5. 'Derivation' of the Schrödinger equation with classical arguments: Classical mechanics as a special case of quantum mechanics

One can 'derive' the Schrödinger equation with entirely classical arguments and equations (Fong, 1962, pp. 45-52). In order to establish the correspondence between classical mechanics and geometrical optics, one has to take into account the principle of least action (a fundamental principle in classical mechanics: a particle is expected to travel along the path which makes *action* minimum), and the principle of least time (the fundamental principle in geometrical optics: light travels along the shortest path).[4] The correspondence is established by setting $\lambda=H/p$, where H is an undetermined proportionality constant. The above relation contains the wavelength and not the frequency. The latter is connected to energy, with one energy corresponding to each frequency. The relation between energy and frequency cannot be derived, so we arbitrarily write $E=H'\nu$ with H' an undermined constant too. Requiring that the velocity of a free particle ($v_p=dE/dp$) must coincide with the group velocity v_g of the equivalent wave packet, leads to the equality $H'=H$. Using the two equations $\lambda=H/p$ and $E=H\nu$, one can derive the general differential equations that $\Psi(r,t)$ and $\psi(r)$ must satisfy, that is the two Schrödinger equations, with the only difference that H will take the place of Planck's constant h. In this manner, a mathematical substitute for classical mechanics is derived, by using only principles and equations of classical mechanics. Instead of solving the Newtonian equations of motion, one has to solve the corresponding wave equation, and obtain a wave packet that moves in space exactly as a classical particle. The only difference is that a wave packet has a finite size, while a classical particle is dimensionless.[5]

The proportionality constant H is of paramount importance. The size of the particle depends on H. Since $p=H/\lambda$, the smaller the H is, the smaller is λ. But λ determines the size of the particle: when λ approaches zero, the size of the wave packet approaches zero and a wave packet can be considered a point. Quantum mechanics is obtained by setting $H=h=6.6256\times10^{-34}$ J s. This constant is extremely small for the macroscopic world, but for the submicroscopic world of atoms, ions, molecules, etc. h is not small. Thus quantum mechanics will be radically different from classical mechanics.

Staying in the analogy between classical mechanics and geometrical optics, we mention another derivation that is based on it and is found for instance in Landau and Lifschiffs (1965) and in Atkins (1970). Finally, for a physical and philosophical discussion for the need for a quantum theory and the differences between quantum mechanics and classical mechanics see Dirac (1958, pp. 1-14). In particular, the probabilistic, and not deterministic character of quantum mechanics is discussed with reference to the polarization of a photon.

CONCLUSIONS AND IMPLICATIONS FOR UNIVERSITY SCIENCE TEACHING

" It is the successes of quantum mechanics in (numerous) applications that justify its basic assumptions and establish its validity. Revolutions and modifications of physical theories may happen in the future, but the new theories will almost certainly have to include quantum mechanics as a special case, and many quantum-mechanical results will last as established parts of our scientific knowledge" (Fong, 1962, p. 47).

The Schrödinger equation is the foundation of quantum mechanics. All the effort in quantum physics and quantum chemistry research lies in formulating approximate solutions to this equation, since exact solutions do not exist except for some simple systems. Therefore, various approximations are in fashion one time or another, and those which are good for today may not be so for tomorrow. One thing however is certain: the Schrödinger equation will still be valid and still be the starting point for any new improvement.

Quantum mechanics rightly then is a strong constituent of the education and training of modern physicists and chemists. It is recognonised, however, that it poses considerable conceptual challenges to students (Tsaparlis, 1993, 1997). From very early (Pauling & Wilson, 1935) it has been realised that "quantum mechanics is essentially mathematical in character, and an understanding of the subject without a thorough knowledge of the mathematical methods involved and the results of their application cannot be obtained." It has even been argued that thinking abilities beyond Piagetian formal operations may be of major importance for an adequate understanding of quantum mechanics (Castro & Fernandez, 1987), including what has been long ago described as quantum logic (Birkhoff & von Neumann, 1936). Finallly, turning our attention to quantum chemistry, we must take into account Coulson's (1974, p. 17) comment that "mathematics is now so central, so much 'inside' that without it cannot hope to understand our chemistry ...".

Science educators argue that to acquire scientific concepts, students have to be exposed to the concepts over an extended period of time. Learning is an evolutionary process that does not occur simultaneously among all students. In addition, instruction must include debates among alternative models and methods (Nussbaum, 1997). The historical approach, together with the variety of current introductions to quantum mechanics, which has been the subject of this work, could serve well the aim of a meaningful learning. To this can be added philosophical-epistemological arguments (Nussbaum, 1997; Baggot, 1992).

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NOTES

1. From $\hat{T}\hat{U}_t = \hat{U}\ \hat{T}$, we get $\hat{T}\ (d\hat{U}_t/dt)\ + (d\hat{T}/dt)\ \hat{U}_t = \hat{U}\ (d\hat{T}/dt)$ (i) $(d\ \hat{U}/dt = 0)$.
    Substituting Ψ with $\hat{T}$ in the Schrödinger form of equation of motion, we have
    $\hat{H}\hat{T}\ = i\hbar(d\hat{T}/dt)$, from which $d\hat{T}/dt = (1/i\hbar)\ \hat{H}\hat{T}$ (ii). With equation (ii), equation (i) gives
    $i\hbar\ (d\hat{U}_t/dt) = \hat{T}^{-1}\hat{U}\ \hat{H}\ \hat{T} - \hat{T}^{-1}\ \hat{H}\ \hat{T}\ \hat{U}_t = \hat{T}^{-1}\ \hat{U}\ \hat{T}\ \hat{H}_t - \hat{T}^{-1}\ \hat{H}\ \hat{T}\ \hat{U}_t = \hat{U}_t\ \hat{H}_t - \hat{H}_t\hat{U}_t$,
    Q.E.D.
2. Distinguish between Schrödinger and Heisenberg dynamical variables: they are connected through a unitary transformation. Thus, all algebraic and functional relationships are the same for both kinds of dynamical variables. For $t = t_o$, $\hat{T} = 1$ and $u_{to} = u$, and only then a Heisenberg variable at time t_o equals the corresponding Schrödinger variable.

3. An introduction to quantum mechanics according to the Heisenberg method is made, for instance, by Landé (1951, pp. 107-150) or Tomonaga (1962). On the other hand, the method of Dirac can be found in his book (Dirac, 1958, pp. 1-135), or in the textbooks by Messiah (1961, pp. 243-340) and by Harris (1975).

4. It can be shown that the principle of least action is equivalent to the newtonian equations of motion. In addition, from the principle of least time, the law of reflection of light and the law of refraction of light (Snell's law) follow (Fong, 1962, pp. 35-39).

5. For a derivation of Newton's second law of motion, with certain approximations, from the time-dependent Schrödinger equation, see Fong (1962, pp. 54-58) or Kemble (1975, pp. 49-51). There is also a parallel between the Schrödinger equation and the second form of the Hamilton-Jacobi equation of classical mechanics.

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